2014-1a

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} \underset{\text{Try to}}{=} \int_{S'} f(z)\,dz$$

$$S' = \{z \in \mathbb{C} : |z| = 1\}$$

$$\int_S f(z)\,dz \underset{\substack{\text{Method} \\ z(\theta)=e^{i\theta} \\ dz = ie^{i\theta}d\theta}}{=} \int_0^{2\pi} \underbrace{f(e^{i\theta})\,ie^{i\theta}}\,d\theta$$

$$\frac{1}{\sqrt{2}-\cos\theta} = \frac{1}{\sqrt{2}-\frac{1}{2}(e^{i\theta}+e^{-i\theta})}$$

$$\therefore f(e^{i\theta}) = \frac{1}{ie^{i\theta}} \cdot \frac{1}{\sqrt{2}-\frac{1}{2}(e^{i\theta}+e^{-i\theta})}$$

Obviously,

$$f(z) = \frac{1}{iz} \frac{1}{\sqrt{2}-\frac{1}{2}(z+\frac{1}{z})} = \frac{1}{i} \frac{1}{\sqrt{2}z-\frac{1}{2}(z^2+1)}$$

$$= \frac{2i}{z^2-2\sqrt{2}z+1} = \frac{2i}{(z-z_1)(z-z_2)}$$

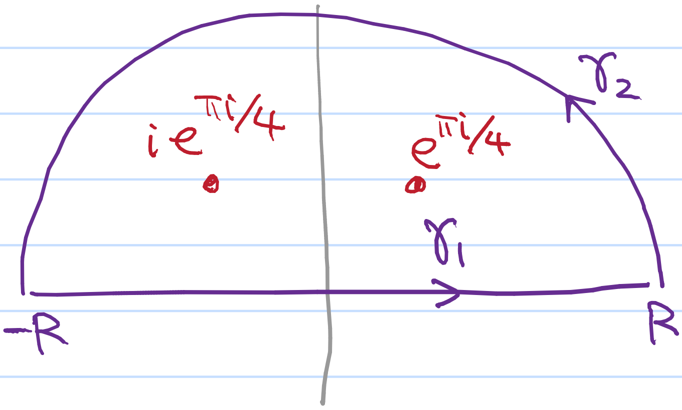
$$z_1 = \sqrt{2}-1, \quad z_2 = \sqrt{2}+1$$

2014-1b

$$\int_0^\infty \frac{dx}{1+x^4} \underset{\text{even function}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

Let $\Gamma = (\gamma_1, \gamma_2)$

$$\int_\Gamma f(z)\,dz$$



$$= 2\pi i \left[\operatorname{Res}\left(f, e^{\pi i/4}\right) + \operatorname{Res}\left(f, ie^{\pi i/4}\right) \right]$$

$$= 2\pi i \left[\frac{1}{4e^{3\pi i/4}} + \frac{1}{4 i^3 e^{3\pi i/4}} \right]$$

$$= 2\pi i \cdot \frac{1}{4}(1+i)\, e^{-3\pi i/4} = \frac{\pi}{\sqrt{2}} \quad \text{indep } R>1$$

Need to argue

$$\left| \int_{\gamma_1} f(z)\,dz \right| \leq \frac{\pi R}{R^4-1} \to 0 \text{ as } R \to \infty$$

2014-1C

$$\int_0^\infty \frac{x\sin(2x)}{x^4+2}\,dx \underset{\substack{\text{even}\\\text{function}}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+2}\,dx$$

Use $\Gamma = (\gamma_1, \gamma_2)$ as in part b.

Work on $\displaystyle\int_\Gamma \frac{ze^{2iz}}{z^4+2}\,dz$

$$= 2\pi i \left[\mathrm{Res}\left(f, 2^{\frac{1}{4}} e^{\pi i/4}\right) + \mathrm{Res}\left(f, 2^{\frac{1}{4}} e^{3\pi i/4}\right)\right]$$

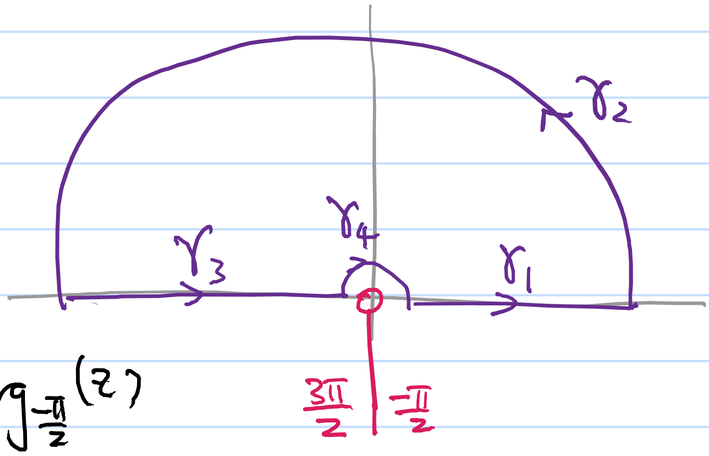
independent of $R > 2^{1/4}$

Need to argue

$$\left|\int_{\gamma_2} \frac{ze^{2iz}}{z^4+2}\,dz\right| \le \frac{R}{R^4-2}\int_0^\pi e^{-2R\sin t} R\,dt \to 0$$

2014-5a

$$\int_0^\infty \frac{\log x}{x^2+4}\,dx$$



Use $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$

and $\mathrm{Log}_{\frac{-\pi}{2}} z = \ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)$

$$\mathrm{Arg}_{\frac{-\pi}{2}}(z) \in \left(\frac{-\pi}{2}, \frac{3\pi}{2}\right)$$

$$\int_{\gamma_1} \frac{\mathrm{Log}_{-\pi/2} z}{z^2+4}\,dz = \int_\delta^R \frac{\log x}{x^2+4}\,dx \to \int_0^\infty \frac{\log x}{x^2+4}\,dx$$

$$\int_{\gamma_3} \frac{\mathrm{Log}_{-\pi/2} z}{z^2+4}\,dz = \int_{-R}^{-\delta} \frac{\log|x| + i(\pi)}{x^2+4}\,dx$$

$$\overset{t=-x}{=} \int_R^\delta \frac{\log t + i(\pi)}{t^2+4}(-dt)$$

$$= \int_\delta^R \frac{\log t + i(\pi)}{t^2+4}\,dt$$

$$\to \int_0^\infty \frac{\log x}{x^2+4}\,dt + i\pi\int_0^\infty \frac{1}{t^2+4}\,dt$$

Need to argue

$$\left|\int_{\gamma_2} \frac{\mathrm{Log}_{-\pi/2} z}{z^2+4}\,dz\right| \leq \int_0^\pi \frac{\log R + it}{R^2-4}\cdot R\,dt$$

$$\to 0 \text{ as } R\to\infty$$

$$\left|\int_{\gamma_4} \frac{\mathrm{Log}_{-\pi/2} z}{z^2+4}\,dz\right| \leq \int_0^\pi \frac{\log\delta + it}{|\delta^2 e^{2it}+4|}\,\delta\,dt \to 0$$